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Interpolation of metric spaces
by
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Errata: In connection with the Cauchy completion we introduce the notation $d_X(x, y)$ and we call it a metric. However, as we shall see in the following example, d_X does not fulfil the triangle inequality. We may notice that in the normed case no trouble will arise.

Example: Let X be the unit circle in the complex plane endowed with the usual metric. As X we choose

$$X = \{z: z \in \mathbb{C}, z = e^{i\theta}, 0 \leq \theta < 2\pi\}.$$

As a metric on X we use

$$d_X(e^{i\theta_1}, e^{i\theta_2}) = |\theta_1 - \theta_2|$$

Since

$$|e^{i\theta_1} - e^{i\theta_2}| \leq |\theta_1 - \theta_2|$$

the inequality (1.1) is fulfilled. It is obvious that $1 \in X$. Let $0 < \theta < \pi/2$. Then

$$d_X(e^{i\theta}, 1) = \theta; \quad d_X(e^{i(2\pi-\theta)}, 1) = \theta; \quad d_X(e^{i\theta}, e^{i(2\pi-\theta)}) = 2\pi - 2\theta.$$

Under these circumstances the triangle inequality cannot be true.

In order to repair the mistake we define a metric $\tilde{d}_X$ on X via

$$(1.3'') \quad \tilde{d}_X(x, y) = \inf \sum_{v=0}^{n-1} d_X(\xi_v, \xi_{v+1})$$

where infimum is taken over all finite sequences $\{\xi_v\}_{v=0}^{n-1}$ in X with $\xi_0 = x$ and $\xi_n = y$.

In all propositions and theorems d_X shall be replaced by $\tilde{d}_X$. In the proofs we need some additional, simple argument.

As an example we look at (4.6). We start from (4.6) in its present form. Let $\{\xi_v\}_{v=0}^{n-1}$ be a sequence in $\tilde{X}_{\theta, q}^K$ with $\xi_0 = x$ and $\xi_n = z$. Then (4.6) yields

$$\sum_{v=0}^{n-1} d(T\xi_v, T\xi_{v+1}; \tilde{Y}_{\theta, q}^K) \leq \omega_0^{1-\theta} \omega_1^\theta \sum_{v=0}^{n-1} d(\xi_v, \xi_{v+1}; \tilde{X}_{\theta, q}^K)$$

The proof is complete if we take infimum of the both members.

In connection with th. 4.1 we notice that condition (4.2) may be omitted.



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0. Introduction.

The study of interpolation spaces has mostly been confined to spaces with an additive structure (cf. Peetre [3], Peetre-Sparr [6]). However in Peetre [4] there is indicated the possibility of defining interpolation spaces between metric spaces. In the present paper we study certain analogues of the K - and J -spaces. In trying to extend to this case what is known in the additive case many complications arise and we have not yet overcome by far all of them. Among the results obtained there is in particular (Section 4) an interpolation theorem for Lipschitz operators - theorems of this kind have earlier been proved by Lions [2], Peetre [4], Tartar [8] - and also (Section 5) a generalization of part of a theorem by Peetre [5] concerning ϵ -capacity.

The subject of this paper was suggested to me by Professor Jaak Peetre. I thank him for great advice and valuable interest in my work.

1. Preparations on metric spaces.

Let X be a non-empty set.

Definition 1.1. By a metric in X we mean a mapping

$d: X \times X \rightarrow R_+ : (x,y) \mapsto d(x,y)$ such that

$$d(x,y) = 0 \iff x = y,$$

$$d(y,x) = d(x,y),$$

$$d(x,y) \leq d(x,z) + d(z,y).$$

Definition 1.2. By a metric space we mean an entity (X,d) consisting of two objects: a non-empty set X and a metric in X .

If there is no danger of misunderstanding, we denote the metric space (X,d) also simply by X . In order to emphasize the dependence on the metric on X we also write $d_X(x,y)$ or sometimes $d(x,y;X)$, instead of just $d(x,y)$.

Definition 1.3. Let X and $\tilde{X}$ be two metric spaces such that $X \subset \tilde{X}$ and such that for some C holds

$$(1.1) \quad d_{\tilde{X}}(x,y) \leq C d_X(x,y), \quad x,y \in X.$$

The Cauchy completion of X relative to $\tilde{X}$, denoted by $\check{X}$, is the set of all $x \in \tilde{X}$ such that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X with

$$(1.2) \quad d_{\tilde{X}}(x_n, x) \rightarrow 0, \quad n \rightarrow \infty; \quad d_X(x_n, x_m) \rightarrow 0, \quad \min(n,m) \rightarrow \infty.$$

(A sequence $\{x_n\}_{n=1}^{\infty}$ satisfying (1.2) will in the sequel be termed approximation to x .) $\check{X}$ becomes a metric space under the metric

$$(1.3) \quad d_{\check{X}}(x,y) = \inf \lim_{n \rightarrow \infty} d_X(x_n, y_n).$$

where the infimum is taken over all approximations $\{x_n\}$ and $\{y_n\}$ to x and y respectively. (cf. Gustavsson [1].)

The positive definiteness of $d_{\check{X}}(x,y)$ follows from (1.1).
Indeed, it is easily seen that

$$(1.4) \quad d_{\check{X}}(x,y) \leq C d_X(x,y), \quad x,y \in \check{X}$$

Note also that obviously

$$(1.5) \quad d_{\check{X}}(x,y) \leq d_X(x,y), \quad x,y \in X$$

and that X (as a set) is dense in $\check{X}$.

2. Definition and some elementary properties of K - and J -spaces.

Let X_0 and X_1 be two sets with non-empty intersection $X_0 \cap X_1$ and both included in a set $\mathcal{X}$. Moreover assume that d_{X_0} , d_{X_1} and $d_{\mathcal{X}}$ are metrics on X_0, X_1 and $\mathcal{X}$ respectively, such that with suitable constants C_i

$$(2.1) \quad d_{\mathcal{X}}(x, y) \leq C_i d_{X_i}(x, y), \quad x, y \in X_i \quad (i = 0, 1).$$

Such an entity we denote by $\vec{X} = \{X_0, X_1\}$ (metric couple).

The following definition as well as def. 2.2 - 2.3 below are due to Peetre [4].

Definition 2.1. For $x, y \in X_0 \cup X_1$ and $t > 0$ we put

$$(2.2) \quad K(t, x, y) = K(t, x, y; \vec{X}) = \inf_{\substack{\varepsilon_v \\ v=0}}^{n-1} t^{\varepsilon_v} d_{\mathcal{X}}(\xi_v, \xi_{v+1})$$

where the infimum is taken over all finite families $\{\xi_v\}_{v=0}^n$, with $\xi_0 = x$, $\xi_n = y$ and ξ_v, ξ_{v+1} both in X_0 or in X_1 ($v = 0, \dots, n-1$), and all finite families $\{\varepsilon_v\}_{v=0}^{n-1}$ with $\varepsilon_v = 0$ or 1.

It is trivial that for t fixed $K(t, x, y)$ is a metric in $X_0 \cup X_1$; the positive definiteness in particular is a consequence of (2.1) (see also (2.7)).

Remark 2.1. Condition (2.1) is essential. If we only assume that $\mathcal{X}$ is a Hausdorff topological space and that X_0 and X_1 are continuously embedded in $\mathcal{X}$, then $K(t, x, y)$ may not be positive definite. (Cf. Peetre [4].) This is seen from

Example 2.1. Let $X_0 = X_1 = \mathcal{X}$ = the set of rational numbers of the form $x = \frac{1}{n}$ ($n = 1, 2, \dots$) and take

$$d_{X_0}(x, y) = |x - y|$$

$$d_{X_1}(x, y) = d_{\mathcal{X}}(x, y) = |\chi(x) - \chi(y)|$$

where χ is any bijection of X onto a dense subset of $\mathbb{R}$. Since X_0 carries the discrete topology, it is clear that the embedding of X_0 into X is continuous. We claim that $K(t, x, y) = 0$ for all t, x, y . Indeed for any $\epsilon > 0$ choose ξ_1 and ξ_2 such that

$$d_{X_1}(x, \xi_1) = |\chi(x) - \chi(\xi_1)| < \epsilon/3t,$$

$$d_{X_1}(\xi_2, y) = |\chi(\xi_2) - \chi(y)| < \epsilon/3t,$$

$$|\xi_1| < \epsilon/6, |\xi_2| < \epsilon/6.$$

Since now

$$d_{X_0}(\xi_1, \xi_2) = |\xi_1 - \xi_2| \leq |\xi_1| + |\xi_2| < \epsilon/6 = \epsilon/3$$

we obtain

$$K(t, x, y) < t \cdot \epsilon/3t + \epsilon/3 + t \cdot \epsilon/3t = \epsilon$$

and the assertion follows.

Definition 2.2. For $x, y \in X_0 \cap X_1$ and $t > 0$ we put

$$(2.3) \quad J(t, x, y) = J(t, x, y; \tilde{X}) = \max(d_{X_0}(x, y), td_{X_1}(x, y)).$$

The following four properties are trivial:

Proposition 2.1. We have for $t > 0$ and $s > 0$

$$(2.4) \quad K(t, x, y) \leq \max(1, \frac{t}{s})K(s, x, y), \quad x, y \in X_0 \cup X_1$$

$$(2.5) \quad J(t, x, y) \leq \max(1, \frac{t}{s})J(s, x, y), \quad x, y \in X_0 \cap X_1$$

$$(2.6) \quad K(t, x, y) \leq \min(1, \frac{t}{s})J(s, x, y), \quad x, y \in X_0 \cap X_1$$

$$(2.7) \quad \min(1, t)d_{\tilde{X}}(x, y) \leq CK(t, x, y), \quad x, y \in X_0 \cup X_1.$$

In the sequel we shall besides (2.1) also assume that the embedding

$$(2.8) \quad (X_0 \cup X_1, d_X) \rightarrow (X_0 \cup X_1, K(1, \dots))$$

is continuous and that X is complete.

Definition 2.3. We set, with $0 < \theta < 1$, $1 \leq q \leq \infty$,

$$\phi_{\theta q}(\phi) = \begin{cases} \left(\int_0^\infty (t^{-\theta} \phi(t))^q dt/t \right)^{1/q}, & 1 \leq q < \infty \\ \sup_{t>0} t^{-\theta} \phi(t), & q = \infty \end{cases}$$

for positive measurable functions $\phi = \phi(t)$. We also set

$$r_{\theta q}(c) = \begin{cases} \left(\sum_{j=-\infty}^\infty (2^{-j\theta} c_j)^q \right)^{1/q}, & 1 \leq q < \infty \\ \sup_j 2^{-j\theta} c_j, & q = \infty \end{cases}$$

for positive sequences $c = \{c_j\}_{j=-\infty}^\infty$.

It is plain that

$$(2.9) \quad \beta(x, y) = \phi_{\theta q}(K(t, x, y))$$

is a metric in $X_0 \cap X_1$.

We can now define the K-spaces:

Definition 2.4. $\vec{X}^K = (X_0, X_1)_{\theta q}^K$ is the Cauchy completion of $(X_0 \cap X_1, \beta)$ relative to (X, d_X) .

Next we define a metric in $X_0 \cap X_1$ by putting

$$(2.10) \quad \alpha(x, y) = \inf r_{\theta q}(c)$$

where the infimum is taken over all sequences $c = \{c_j\}_{j=-\infty}^\infty$ of the type

$$(2.11) \quad c_j = \sum_{k=0}^\infty j(2^k, \xi_k, \xi_{k+1})$$

where $\{\xi_\nu\}_{\nu=0}^n$ is any finite family in $X_0 \cap X_1$ such that $\xi_0 = x$, $\xi_n = y$ and $\{k_\nu\}_{\nu=0}^{n-1}$ any finite family of integers. The positive definiteness again is a consequence of (2.1) (see also (2.12)).

Then we can define the J-spaces too:

Definition 2.5. $\tilde{X}_{\theta q}^J = (X_0, X_1)_{\theta q}^J$ is the Cauchy completion of $(X_0 \cap X_1, \alpha)$ relative to (X, d_X) .

Proposition 2.2. We have

$$(2.12) \quad d_X(x, y) \leq C d_E(x, y), \quad x, y \in E$$

$$(2.13) \quad K(t, x, y) \leq C t^\theta d_E(x, y), \quad x, y \in (X_0 \cup X_1) \cap E$$

$$(2.14) \quad d_E(x, y) \leq C t^{-\theta} J(t, x, y), \quad x, y \in X_0 \cap X_1$$

where $E = \tilde{X}_{\theta q}^K$ or $\tilde{X}_{\theta q}^J$.

(Note that (2.14) is equivalent to

$$(2.15) \quad d_E(x, y) \leq C [d_{X_0}(x, y)]^{1-\theta} [d_{X_1}(x, y)]^\theta, \quad x, y \in X_0 \cap X_1.$$

Proof: By (2.7) it is plain that (2.12) follows from (2.13). We therefore need only to prove (2.13) and (2.14).

We start by proving (2.13) for $E = \tilde{X}_{\theta q}^K$. First let $x, y \in X_0 \cap X_1$. Writing (2.4) as

$$K(t, x, y) \min(1, \frac{s}{t}) \leq K(s, x, y).$$

we then get

$$K(t, x, y) \phi_{\theta q}[\min(1, \frac{s}{t})] \leq \phi[K(s, x, y)] = \beta(x, y)$$

Since $\phi_{\theta q}[\min(1, \frac{s}{t})] = C^{-1} t^{-\theta}$ we get

$$(2.16) \quad K(t, x, y) \leq C t^\theta \beta(x, y).$$

Now (2.13) follows from (2.16) by a continuity argument. Indeed, if $x, y \in (X_0 \cup X_1) \cap E$ choose approximations $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ to x and y respectively. In view of the continuity of (2.8) we get from (2.16) with x, y replaced by x_n, y_n :

$$K(t, x, y) = \lim_{n \rightarrow \infty} K(t, x_n, y_n) \leq Ct^\theta \lim_{n \rightarrow \infty} \beta(x_n, y_n)$$

This clearly implies (2.13).

For the proof of (2.13) for $E = X_{\theta q}^J$ we first again take $x \in X_0 \cap X_1$. Let $\{\xi_v\}_{v=0}^n$ and $\{k_v\}_{v=0}^{n-1}$ be two of the families entering in the definition of (2.10). By the triangle inequality and (2.6) we obtain

$$\begin{aligned} (2.17) \quad K(t, x, y) &\leq \sum_{v=0}^{n-1} K(t, \xi_v, \xi_{v+1}) \leq \\ &\leq \sum_{v=0}^{n-1} \min(1, \frac{t}{k_v}) J(2^{\frac{k_v}{2}}, \xi_v, \xi_{v+1}) = \sum_{i=-\infty}^{\infty} \min(1, \frac{1}{2^i}) c_i \end{aligned}$$

with c_i given by (2.11). Hölder's inequality then gives

$$K(t, x, y) \leq \Gamma_{-\theta, q}(\min(1, \frac{t}{2^i})) \Gamma_{\theta q}(c) \leq Ct^\theta \Gamma_{\theta q}(c)$$

which clearly implies

$$(2.18) \quad K(t, x, y) \leq Ct^\theta \alpha(x, y)$$

From (2.18) follows again (2.13), by a similar density argument.

Next we prove (2.14) with $E = X_{\theta q}^K$. From (2.6) follows at once

$$\alpha(x, y) \leq Ct^{-\theta} J(t, x, y),$$

which clearly implies (2.14), by (1.5).

Finally we prove (2.14) with $E = \tilde{X}_{\theta q}^J$. In the definition of (2.10) take $\xi_0 = x$, $\xi_1 = x$, $k_0 = i$ ($n = 1$). Then we get at once

$$\alpha(x, y) \leq Ct^{-\theta} J(t, x, y)$$

with $t = 2^i$ and then in view of (2.5) for any t . This clearly again implies (2.14). $\#$

Proposition 2.3. $\tilde{X}_{\theta q}^K$ is complete.

For the proof of prop. 2.3 we need a lemma:

Lemma 2.1. If $x, y \in X_0 \cap X_1$ then

$$(2.19) \quad \beta(x, y) = d(x, y; \tilde{X}_{\theta q}^K).$$

Proof: Let $x, y \in X_0 \cap X_1$ and choose approximations $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ to x and y respectively. Using (2.4) and the continuity of (2.8) we see that

$$K(t, x_n, x) \rightarrow 0, \quad K(t, y_n, y) \rightarrow 0$$

Thus by the triangle inequality

$$K(t, x_n, y_n) \rightarrow K(t, x, y).$$

If $q < \infty$ we then get by Fatou's lemma

$$\begin{aligned} \beta(x, y) &= \left(\int_0^{\infty} (t^{-\theta} K(t, x, y))^q \frac{dt}{t} \right)^{1/q} \leq \\ &\leq \lim_{n \rightarrow \infty} \left(\int_0^{\infty} (t^{-\theta} K(t, x_n, y_n))^q \frac{dt}{t} \right)^{1/q} = \lim_{n \rightarrow \infty} \beta(x_n, y_n) \end{aligned}$$

Thus

$$\beta(x, y) \leq d(x, y; \tilde{X}_{\theta q}^K).$$

Since the converse inequality is trivial (see (1.5)), (2.19) follows. If $q = \infty$ a trivial modification of this argument will do. $\#$

We can now provide the

Proof of prop. 2.3.: Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\hat{X}_{\theta q}^K$. Since $X_0 \cap X_1$ is dense in $X_{\theta q}^K$, it is no restriction to assume that $x_n \in X_0 \cap X_1$. Since X is complete we can find $x \in X$ such that

$$d_X(x_n, x) \rightarrow 0, \quad n \rightarrow \infty.$$

On the other hand by lemma 2.1

$$\beta(x_n, x_m) \rightarrow 0, \quad \min(n, m) \rightarrow \infty.$$

In other words, x belongs to the Cauchy completion of $(X_0 \cap X_1, \beta)$ relative to (X, d_X) or $x \in \hat{X}_{\theta q}^K$ by def. 2.4. For $\epsilon > 0$ choose now N such that

$$\beta(x_n, x_m) < \epsilon \quad \text{if } \min(m, n) \geq N$$

Then we get

$$d(x, x_n; \hat{X}_{\theta q}^K) \leq \epsilon \quad \text{if } n \geq N$$

and we see that $\{x_n\}_{n=1}^{\infty}$ is convergent in $\hat{X}_{\theta q}^K$. $\#$

Remark 2.2. Prop. 2.3 together with lemma 2.1 tell us that $\hat{X}_{\theta q}^K$ may be considered as a realization of the abstract completion of $(X_0 \cap X_1, \beta)$. Thus our definition is consistent with the one suggested in Peetre [4]. It follows also that if $q < \infty$ we obtain in the additive case the same spaces as usual, if $q = \infty$ this is not the case (cf. Peetre [3]).

3. Relations between K- and J-spaces.

In the additive case the K- and J-methods are equivalent. We do not know if this is true in general in the metric case. We list here the following fragmentary results.

Theorem 3.1. We have

$$(3.1) \quad \tilde{X}_{\theta q}^J \subset \tilde{X}_{\theta q}^K.$$

$$(3.2) \quad d(x, y; \tilde{X}_{\theta q}^K) \leq Cd(x, y; \tilde{X}_{\theta q}^J), \quad x, y \in \tilde{X}_{\theta q}^J$$

Proof: It is clear that it suffices to prove

$$(3.3) \quad \beta(x, y) \leq C\alpha(x, y), \quad x, y \in X_0 \cap X_1.$$

We rewrite (2.15) (with $t = 2^j$) as

$$K(2^j, x, y) \leq \sum_{i=-\infty}^{\infty} \min(1, 2^{-i}) c_{i+j}$$

This yields

$$\begin{aligned} r_{\theta q}(K(2^j, x, y)) &\leq \sum_{j=-\infty}^{\infty} \min(1, 2^{-i}) r_{\theta q}(c_{i+j}) \leq \\ &\leq C \sum_{i=-\infty}^{\infty} \min(1, 2^{-i}) 2^{i\theta} r_{\theta q}(c) \leq Cr_{\theta q}(c) \end{aligned}$$

Since by (2.4)

$$(3.4) \quad \phi_{\theta q}(K(t, x, y)) \leq Cr_{\theta q}(K(2^j, x, y))$$

we get

$$\beta(x, y) \leq Cr_{\theta q}(c)$$

which implies (3.3), in view of (2.10). $\#$

Next, imitating Sparr [7], we make the following definition

Definition 3.1. We say that $\tilde{X} = \{X_0, X_1\}$ fulfils condition H if for every $x, y \in X_0 \cap X_1$ we can find two families $\{\xi_v\}_{v=0}^n$ and $\{k_v\}_{v=0}^{n+1}$ of the type entering in the definition of (2.10) such that for some constant γ holds

$$(3.5) \quad c_j \leq \gamma K(2^j, x, y),$$

with c_j as in (2.11).

It is easy to give plenty of couples satisfying H (see Sparr [7]).

We can now prove a result in the opposite sense.

Theorem 3.2. Assume that $\tilde{X}$ fulfils condition H. Then we have

$$(3.6) \quad \tilde{X}_{\theta q}^I \subset \tilde{X}_{\theta q}^J$$

$$(3.7) \quad d(x, y; \tilde{X}_{\theta q}^J) \leq Cd(x, y; \tilde{X}_{\theta q}^K), \quad x, y \in \tilde{X}_{\theta q}^K.$$

Proof: This time we have to prove that

$$(3.8) \quad \alpha(x, y) \leq C\beta(x, y), \quad x, y \in X_0 \cap X_1.$$

Indeed from (3.5) immediately follows

$$\alpha(x, y) \leq \Gamma(c) \leq \gamma \Gamma(K(2^i, x, y))$$

and, since the converse of (3.4) also holds, (3.8) follows. $\#$

Corollary 3.1. In the same assumptions we have thus

$$(3.9) \quad \tilde{X}_{\theta q}^K = \tilde{X}_{\theta q}^J. \quad \#$$

4. Interpolation of Lipschitz operators.

In this Section we consider two metric couples $\tilde{X} = \{X_0, X_1\}$ and $\tilde{Y} = \{Y_0, Y_1\}$ satisfying the same assumptions as $\tilde{X}$ in Section 1 (i.e. (2.4), continuity of (2.8), completeness of $\tilde{X}$).

Theorem 4.1. If the mapping, T has the following four properties

$$(4.1) \quad T : X \rightarrow Y \quad (\text{where } Y \text{ is the space in which } Y_0 \text{ and } Y_1 \text{ are embedded}),$$

$$(4.2) \quad \lim_{n \rightarrow \infty} d_X(x_n, x) = 0, \quad \lim_{n \rightarrow \infty} d_Y(Tx_n, y) = 0 \implies Tx = y,$$

$$(4.3) \quad T|_{X_i} : X_i \rightarrow Y_i \quad (i = 0, 1),$$

$$(4.4) \quad d_{Y_i}(Tx, Tz) \leq \omega_i d_{X_i}(x, z), \quad x, z \in X_i \quad (i = 0, 1)$$

then

$$(4.5) \quad T|_{X_{\theta q}^K} : X_{\theta q}^K \rightarrow Y_{\theta q}^K$$

$$(4.6) \quad d(Tx, Tz; Y_{\theta q}^K) \leq \omega_0^{1-\theta} \omega_1^{\theta} d(x, z; X_{\theta q}^K), \quad x, z \in X_{\theta q}^K,$$

Proof: First we note that (4.3) and (4.4) imply

$$K(t; Tx, Tz; \tilde{Y}) \leq \omega_0^{1-\theta} \omega_1^{\theta} K(t; x, z; \tilde{X}), \quad x, z \in X_0 \cup X_1$$

which clearly implies

$$(4.7) \quad \beta_Y(Tx, Tz) \leq \omega_0^{1-\theta} \omega_1^{\theta} \beta_X(x, z), \quad x, z \in X_0 \cap X_1$$

with β_X and β_Y defined as in (2.9). Let now $x \in X_{\theta q}^K$ and choose an approximation $\{x_n\}_{n=1}^{\infty}$ to x . Since

$$\beta_X(x_n, x_m) \rightarrow 0, \quad \min(n, m) \rightarrow \infty$$

follows from (4.7) that

$$\beta_Y(Tx_n, Tx_m) \rightarrow 0, \quad \min(n, m) \rightarrow \infty.$$

Since Y is complete there exists an element $y \in Y$ such that

$$(4.8) \quad d_{Y_1}(Tx_n, y) \rightarrow 0$$

By definition $y \in \tilde{Y}_{\theta q}^K$. Applying (4.2) we see that $y = Tx$.

Thus we have proved (4.5). The proof of (4.6) follows now readily from (4.7) by a new approximation argument. #

Theorems of this kind have been given by Lions [2], Peetre [4] and Tartar [8].

Corollary 4.1. If $X_0 \subset X_1$ and

$$d_{X_1}(x, y) \leq Cd_{X_0}(x, y), \quad x, y \in X_0$$

and T is a mapping such that (4.1), (4.3) and (4.4) are true, then (4.5) and (4.6) are true.

Proof: Let

$$\lim_{n \rightarrow \infty} d_X(x_n, x) = 0, \quad \lim_{n \rightarrow \infty} d_Y(Tx_n, y) = 0.$$

Since

$$\begin{aligned} d_Y(Tx_n, Tx) &\leq CK(1, Tx_n, Tx; \tilde{Y}) \leq Cd_{Y_1}(Tx_n, Tx) \leq \\ &\leq Cd_{X_1}(Tx_n, Tx) \leq C_2K(a, x_n, x; \tilde{X}), \end{aligned}$$

we see that $y = Tx$. Thus (4.2) is automatically fulfilled in this case and we can apply th. 4.1. #

(Cf. remark 1 in Tartar [8] and the proposition in Tartar [9].)

Corollary 4.2. If $X_i = Y_i$ ($i = 0, 1$), $X = Y$ and $\omega_0^{1-\theta} \omega_1 < 1$ then T has a unique fixed point in $\tilde{X}_{\theta q}^K$.

Proof: Since $\tilde{X}_{\theta q}^K$ is complete by prop. 2.3, the Banach contraction principle can be applied. #

Finally we state an analogue of th. 4.1 for J-spaces.

Theorem 4.2. If T is a mapping such that (4.1) - (4.4) are fulfilled. Then we have

$$(4.8) \quad T : \tilde{X}_{\theta q}^J \rightarrow \tilde{Y}_{\theta q}^J$$

$$(4.9) \quad d_{\tilde{Y}_{\theta q}^J}^J(Tx, Tz) \leq 2^{\theta} \omega_0^{1-\theta} \omega_1^{\theta} d_{\tilde{X}_{\theta q}^J}^J(x, z), \quad x, z \in \tilde{X}_{\theta q}^J$$

Proof: In place of (4.7) we now have

$$(4.10) \quad \alpha_{\tilde{Y}}^J(Tx, Tz) \leq 2^{\theta} \omega_0^{1-\theta} \omega_1^{\theta} \alpha_{\tilde{X}}^J(x, z), \quad x, z \in X_0 \cap X_1,$$

with $\alpha_{\tilde{X}}^J$ and $\alpha_{\tilde{Y}}^J$ defined as in (2.10). The rest of the proof is quite parallel to that of th. 4.1 and will be omitted. $\square$

Remark 4.1. The factor 2^{θ} in (4.9) and (4.10) of course appears, because we have used "base 2" in the definition of J-spaces.

5. ϵ -capacity and interpolation spaces.

In this Section we shall generalize part of the theorem in Peetre [5]. We recall

Definition 5.1. Let Y be a metric space and X a subspace. We put

$$M(\epsilon) = \sup\{M \mid \exists x_1, \dots, x_M \in X : \inf_{\substack{i,j=1,\dots,M \\ i \neq j}} d_Y(x_i, x_j) > \epsilon\}$$

Then $\log_2 M(\epsilon)$ is called the ϵ -capacity of X relative to Y .

Let $\tilde{X} = \{X_0, X_1\}$ be a metric couple such that $X_1 \subset X_0$ with

$$d_{X_0}(x, y) \leq c \cdot d_{X_1}(x, y), \quad x, y \in X_1.$$

Moreover, let $E = \tilde{X}_{\theta q}^K$ or $\tilde{X}_{\theta q}^J$. We now make a variation of def. 5.1. More precisely, for $y \in X_1$ we put:

$$(5.1) \quad M_y(\epsilon; E, X_1) = \sup\{M \mid \exists x_1, \dots, x_M \in X_1 : \sup_{i=1, \dots, M} d_{X_1}(x_i, y) \leq 1, \\ \inf_{\substack{i,j=1, \dots, M \\ i \neq j}} d_E(x_i, x_j) > \epsilon\},$$

$$(5.2) \quad M_y(\epsilon; X_0, X_1) = \sup\{M \mid \exists x_1, \dots, x_M \in X_1 : \sup_{i=1, \dots, M} d_{X_1}(x_i, y) \leq 1, \\ \inf_{\substack{i,j=1, \dots, M \\ i \neq j}} d_{X_0}(x_i, x_j) > \epsilon\}.$$

Now we can state the following

Theorem 5.1. We have for $0 < \theta < 1$

$$(5.3) \quad M_y(\epsilon; E, X_1) \leq M_y\left(\mu\left(\frac{\epsilon}{C}\right)^{\frac{1}{1-\theta}}, X_0, X_1\right)$$

where C is the constant in (2.15) and μ depends only upon θ .

Proof: Choose $x_1, \dots, x_M$ in X_1 such that $M = M_y(\varepsilon; E, X_1)$ and

$$\sup_{i=1, \dots, M} d_{X_1}(x_i, y) \leq 1, \quad \inf_{\substack{i, j=1, \dots, M \\ i \neq j}} d_E(x_i, x_j) > \varepsilon.$$

Using (2.15) and the triangle inequality we get

$$\varepsilon < \inf d_E(x_i, x_j) \leq C(d_{X_0}(x_i, x_j))^{1-\theta} 2^\theta$$

Thus

$$2^{\frac{\theta}{1-\theta}} \left(\frac{\varepsilon}{C}\right)^{1-\theta} \leq \inf d_{X_0}(x_i, x_j)$$

and we have (5.3) with $\mu = 2^{\frac{\theta}{1-\theta}}$. ~~††~~

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